

ELIMINATE OBSTRUCTIONS: CURVES ON A 3-FOLD

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ABSTRACT. By using higher K-theory, we reinterpret and generalize an idea on eliminating obstructions of deformation of cycles, which is known to Mark Green and Phillip Griffiths [3] and TingFai Ng [5].

As an application, we show how to eliminate obstructions of deformation of curves on a 3-fold in an explicit way. This answers affirmatively an open question by TingFai Ng [5].

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1. INTRODUCTION: NG'S QUESTION

Let X be a smooth projective variety over a field k of characteristic 0. For $Y \subset X$ a subscheme of codimension p , Y can be considered as an element of the Hilbert scheme $\mathrm{Hilb}^p(X)$ and the Zariski tangent space $T_Y \mathrm{Hilb}(X)$ can be identified with $H^0(N_{Y/X})$. It is well-known that $\mathrm{Hilb}^p(X)$ may be nonreduced at Y . In other words, let Y' be a first order lifting of Y in $X[\varepsilon]/(\varepsilon^2)$, the lifting of Y' to $X[\varepsilon]/(\varepsilon^3)$ may be obstructed.

However, Green-Griffiths predicts that we can eliminate obstructions in their program [3], by considering Y as a **cycle**. That is, instead of considering Y as an element of $\mathrm{Hilb}^p(X)$, considering Y as an element of the cycles class group $Z^p(X)$ can eliminate obstructions. For $p = 1$, Green-Griffiths' idea was realized by TingFai Ng in his Ph.D thesis [5]:

Theorem 1.1. [5] *The divisor class group $Z^1(X)$ is smooth.*

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In down-to-earth terms, Ng's theorem says the following. For $Y \in Z^1(X)$, let Y' be a first order lifting of Y in $X[\varepsilon]/(\varepsilon^2)$, then we can lift Y' (as a cycle) to $X[\varepsilon]/(\varepsilon^3)$ and then lift it to higher order successively. Ng's method is to use Bloch's *semi-regularity* map and we sketch it briefly as follows.

For Y a locally complete intersection, Bloch constructs *semi-regularity* map in [2],

$$\pi : H^1(Y, N_{Y/X}) \rightarrow H^{p+1}(X, \Omega_{X/k}^{p-1}).$$

In particular, for $p = 1$, this map $\pi : H^1(Y, N_{Y/X}) \rightarrow H^2(O_X)$ agrees with the boundary map in the long exact sequence

$$\cdots \rightarrow H^1(O_X(Y)) \rightarrow H^1(Y, N_{Y/X}) \rightarrow H^2(O_X) \rightarrow \cdots,$$

associated to the short exact sequence:

$$0 \rightarrow O_X \rightarrow O_X(Y) \rightarrow N_{Y/X} \rightarrow 0.$$

Let W be an ample divisor such that $H^1(O_X(Y + W)) = 0$. Since the subscheme $Y \cup W$ is still a locally complete intersection, we have the *semi-regularity* map $\pi : H^1(Y \cup W, N_{Y \cup W/X}) \rightarrow H^2(O_X)$, which agrees with the boundary map in the long exact sequence

$$\cdots \rightarrow H^1(O_X(Y + W)) \rightarrow H^1(Y \cup W, N_{Y \cup W/X}) \rightarrow H^2(O_X) \rightarrow \cdots.$$

Let $(Y \cup W)'$ denote the first order lifting of $Y \cup W$. Since $H^1(O_X(Y + W)) = 0$, the kernel of π is 0. In other words, $Y \cup W$ is *semi-regularity* in X , so according to Theorem 7.3 of [2], the Hilbert scheme $\text{Hilb}(X)$ is smooth at the point corresponding to $Y \cup W$. By infinitesimal lifting property, we can lift $(Y \cup W)'$ to a second order lifting in $X[\varepsilon]/(\varepsilon^3)$. In other words, the obstruction of lifting $(Y \cup W)'$ to a second order lifting in $X[\varepsilon]/(\varepsilon^3)$ vanishes.

As a cycle, Y can be formally written as

$$Y = (Y + W) - W$$

and then we may write Y' , a first order lifting of Y , as

$$Y' = (Y + W)' - W',$$

where W' is a first order lifting of W in $X[\varepsilon]/(\varepsilon^2)$. For our purpose, in order to avoid bringing new obstructions, we fix W , that is, we take $W' = W \subset X[\varepsilon]/(\varepsilon^2)$ and W' can be lifted to $W'' = W \subset X[\varepsilon]/(\varepsilon^3)$. Since both W' and $(Y \cup W)'$ can be lifted to a second order in $X[\varepsilon]/(\varepsilon^3)$, so does Y' .

Idea: W helps to eliminate obstructions, without introducing new obstructions.

For related discussions and intuitive pictures, see Green-Griffiths [3](page 188-189). In Section 1.5 of [5], TingFai Ng asks whether we can extend the above method beyond divisor case, e.g. , curves on a 3-fold X .

Suppose $C' \subset X[\varepsilon]/(\varepsilon^2)$ is a first order lifting of a curve C on a 3-fold X . Such a lifting C' corresponds to a vector field in $H^0(C, N)$. While the lifting of C' to $X[\varepsilon]/(\varepsilon^3)$ may be obstructed(as a subscheme), one can ask whether a lift of C' as a **cycle** always exists:

Question 1.2. [5] *Given a smooth ¹ closed curve C in a 3-fold X and a normal vector field v , we wish to know whether it is always possible to find a nodal curve \tilde{C} in X , of which C is a component, (i.e. $\tilde{C} = C \cup D$ for some residue curve D) and a normal vector field \tilde{v} on \tilde{C} such that*

1. $\tilde{v}|_C = v$,
2. *the first order deformation given by (\tilde{C}, \tilde{v}) extends to second order, and*
3. *the first order deformation given by (D, v') extends to second order.*

Cycle-theoretically, we have $(C, v) = (\tilde{C}, \tilde{v}) - (D, v')$. So we are asking whether (C, v) as a first order deformation of cycles always extends to second order.

The key to answer Ng's question is to interpret it, especially the word **Cycle-theoretically**, in an appropriate way. For this purpose, we reformulate Ng's question in the framework of [8] and answer it affirmatively.

Remark 1.3. *A similar question on eliminating obstructions for X of any dimension and Y of any codimension has been asked by Mark Green and Phillip Griffiths [3], which has been reformulated and has been answered affirmatively in [8].*

We remark that Ng's question above and its reformulation in Question 2.4 below, is different from Green – Griffiths' question on obstruction issues reformulated in [8]. That's mainly because we don't know whether the map μ in Definition 2.2 is surjective or not. The author learned this subtlety from Spencer Bloch.

Notations and conventions.

(1). K-theory used in this note will be Thomason-Trobaugh non-connective K-theory, if not stated otherwise. For any abelian group M , $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. $(a, b)^T$ denotes the transpose of (a, b) .

¹We will remove this hypothesis later in Question 2.4.

(2). X_j denotes the j -th **trivial** infinitesimal deformation of X , i.e., $X_j = X \times_k \text{Spec}(k[\varepsilon]/\varepsilon^{j+1})$. In particular, $X_0 = X$, $X_1 = X[\varepsilon]/(\varepsilon^2)$, and $X_2 = X[\varepsilon]/(\varepsilon^3)$.

To fix notations, $D^{\text{perf}}(X_j)$ denotes the derived category obtained from the exact category of perfect complex on X_j and $\mathcal{L}_{(i)}(X_j)$ is defined to be

$$\mathcal{L}_{(i)}(X_j) := \{E \in D^{\text{perf}}(X_j) \mid \text{codim}_{\text{Krull}}(\text{supph}(E)) \geq -i\}.$$

Let $(\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#}$ denote the idempotent completion of the Verdier quotient $\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j)$,

Theorem 1.4. [1] *For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#} \simeq \bigsqcup_{x_j \in X_j^{(-i)}} D_{x_j}^{\text{perf}}(X_j)$$

between the idempotent completion of the quotient $\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j)$ and the coproduct over $x_j \in X_j^{(-i)}$ of the derived category of perfect complexes of O_{X_j, x_j} -modules with homology supported on the closed point $x_j \in \text{Spec}(O_{X_j, x_j})$. Consequently, localization induces an isomorphism

$$K_0((\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#}) \simeq \bigoplus_{x_j \in X_j^{(-i)}} K_0(O_{X_j, x_j} \text{ on } x_j).$$

2. REFORMULATE NG'S QUESTION

Let X be a nonsingular projective 3-fold over a field k of characteristic 0 and let $Y \subset X$ be a curve(not necessarily locally complete intersection or smooth)with generic point y . For a point $x \in Y \subset X$ with local uniformizer f, g, h , the local ring $O_{X, x}$ is a regular local ring of dimension 3 and the maximal ideal $m_{X, x}$ is generated by a regular sequence f, g, h . To fix notations, we further assume Y is generically defined by f and g . So the local ring $O_{X, y} = (O_{X, x})_{(f, g)}$ is a regular local ring of dimension 2 and the maximal ideal $m_{X, y}$ is generated by the regular sequence f, g . Since $h \notin (f, g)$, h^{-1} exists in $O_{X, y}$.

Let ε be a nilpotent satisfying $\varepsilon^2 = 0$ in this section. Let $Y' \subset X[\varepsilon]$ be a first order infinitesimal deformation of Y , that is, Y' is flat over $\text{Spec}(k[\varepsilon])$ and $Y' \otimes_{k[\varepsilon]} k \cong Y$. While Y' is not necessary to be a trivial deformation, it is generically trivial. Let $\mathcal{I}_{Y'}$ be the ideal sheaf of Y' , the localization at the generic point $(\mathcal{I}_{Y'})_y = (f + \varepsilon f_1, g + \varepsilon g_1)$, where $f_1, g_1 \in O_{X, y}$. See [9] for related discussions if necessary.

We keep the notations in [9] and use $F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1)$ to denote the Koszul complex associated to the regular sequence $f + \varepsilon f_1, g + \varepsilon g_1$, which is a resolution of $O_{X,y}[\varepsilon]/(f + \varepsilon f_1, g + \varepsilon g_1)$:

$$0 \rightarrow O_{X,y}[\varepsilon] \xrightarrow{(g+\varepsilon g_1, -f-\varepsilon f_1)^T} O_{X,y}[\varepsilon]^{\oplus 2} \xrightarrow{(f+\varepsilon f_1, g+\varepsilon g_1)} O_{X,y}[\varepsilon].$$

Recall that Milnor K-group with support is rationally defined as certain eigenspaces of K-groups in [7],

$$\begin{aligned} K_0^M(O_{X,y} \text{ on } y) &:= K_0^{(2)}(O_{X,y} \text{ on } y)_{\mathbb{Q}}, \\ K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) &:= K_0^{(2)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}. \end{aligned}$$

Lemma 2.1.

$$\begin{aligned} K_0^M(O_{X,y} \text{ on } y) &= K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}}, \\ K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) &= K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}. \end{aligned}$$

Proof. According to Riemann-Roch without denominator [6],

$$K_0^{(2)}(O_{X,y} \text{ on } y)_{\mathbb{Q}} \cong K_0^{(0)}(k(y))_{\mathbb{Q}},$$

where $k(y)$ is the residue field. This forces $K_0^{(j)}(O_{X,y} \text{ on } y)_{\mathbb{Q}} = 0$, except for $j = 2$. So we have

$$K_0^M(O_{X,y} \text{ on } y) = K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

Since $K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0(O_{X,y} \text{ on } y)_{\mathbb{Q}} \oplus H_y^2(\Omega_{X/\mathbb{Q}}^1)$, one can check that $K_0^{(j)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = 0$, except for $j = 2$. That is,

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}.$$

□

Definition 2.2. [9] We define a map $\mu : H^0(Y, \mathcal{N}_{Y/X}) \rightarrow K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$ as follows:

$$\begin{aligned} \mu : H^0(Y, \mathcal{N}_{Y/X}) &\rightarrow K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \\ Y' &\longrightarrow F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1). \end{aligned}$$

Now, we recall Milnor K-theoretic cycles:

Definition 2.3. [7, 8] Let X be a nonsingular projective 3-fold over a field k of characteristic 0, for $i = 0, 1, 2$, the 2nd Milnor K-theoretic cycles on X_i are defined as follows:

$$Z_2^M(D^{\text{Perf}}(X_i)) = \text{Ker}(d_{1,X_i}^{2,-2}),$$

where $d_{1,X_i}^{2,-2}$ are the differentials in Theorem 3.1 and Theorem 3.3 below.

Now, we are ready to rewrite Ng's question as follows:

Question 2.4. [5] *Let X be a nonsingular projective 3-fold over a field k of characteristic 0. Given a curve Y in X and a first order infinitesimal deformation Y' of Y , let $\mu(Y') \in K_0(O_{X,y}[\varepsilon])$ on $y[\varepsilon]$ denote the image of Y' under the map μ in Definition 2.2. Is it always possible to find an element $\gamma \in Z_2^M(D^{\text{Perf}}(X))$ such that $\gamma = \mu(Y) + \mu(Z)$ ², for some curve $Z \subset X$ and a first order deformation γ' of γ , in the sense of [8] (Definition 2.10), such that*

1. $\gamma' = \mu(Y') + \mu(Z') \in Z_2^M(D^{\text{Perf}}(X_1))$, with Z' a first order infinitesimal deformation of Z ,

2. the first order deformation γ' extends to second order $\gamma'' \in Z_2^M(D^{\text{Perf}}(X_2))$, and

3. the first order deformation given by $\mu(Z')$ extends to second order.

Cycle-theoretically, we have $\mu(Y') = \gamma' - \mu(Z')$. So we are asking whether $\mu(Y')$ as a first order deformation of cycles always extends to second order.

Definition 2.5. *The ideal $(h, g) \subset O_{X,x}$ defines another curve Z on X :*

$$Z := \text{Spec}(O_{X,x}/(h, g)).$$

We fix the notation Z and use z to denote the generic point of Z . Then $O_{X,z} = (O_{X,x})_{(h,g)}$. Since $f \notin (h, g)$, f^{-1} exists in $O_{X,z}$.

3. ANSWER NG'S QUESTION

Theorem 3.1. [7] *For X a nonsingular projective 3-fold over a field k of characteristic 0 and for $q = 2$ and $j = 1$ in Theorem 3.14 of [7], we have the following commutative diagram (For $w_i = x_i, y_i, z_i$ in the diagram below with $i = 0^3$ and 1, we have $K_*^M(O_{X_i, w_i} \text{ on } w_i) \cong K_*(O_{X_i, w_i} \text{ on } w_i)_{\mathbb{Q}}$. This may be explained (in a similar way) by Lemma 2.1. We omit the subscript \mathbb{Q} in the diagram below):*

² $\mu(Y)$ and $\mu(Z)$ are defined in Definition 2.2. E.g., to define $\mu(Y)$, we take $f_1 = g_1 = 0$.

³By convention, $w_0 = w$ and $X_0 = X$.

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{k(X)/\mathbb{Q}}^1 & \xleftarrow{\text{Chern}} & K_2^M(k(X)[\varepsilon]) & \xrightarrow{\varepsilon=0} & K_2^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{z \in X^{(1)}} H_z^1(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow{\text{Chern}} & \bigoplus_{z[\varepsilon] \in X[\varepsilon]^{(1)}} K_1(O_{X,z}[\varepsilon] \text{ on } z[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{z \in X^{(1)}} K_1(O_{X,z} \text{ on } z) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(2)}} H_y^2(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow{\text{Chern}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{y \in X^{(2)}} K_0(O_{X,y} \text{ on } y) \\
\downarrow \partial_1^{2,-2} & & \downarrow d_{1,X_1}^{2,-2} & & \downarrow d_{1,X}^{2,-2} \\
\bigoplus_{x \in X^{(3)}} H_x^3(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow{\text{Chern}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(3)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) & \xrightarrow{\varepsilon=0} & \bigoplus_{x \in X^{(2)}} K_{-1}(O_{X,x} \text{ on } x) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

For simplicity, we assume $g_1 = 0$ in Definition 2.2 and consider $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1, g)$ in the following, where w_1 is an arbitrary element of $O_{X,x}$ and $\varepsilon^2 = 0$. The Koszul resolution of $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1, g)$,

$$L' \bullet : 0 \rightarrow O_{X,x}[\varepsilon] \xrightarrow{(g, -(fh + \varepsilon w_1))^T} O_{X,x}[\varepsilon]^{\oplus 2} \xrightarrow{(fh + \varepsilon w_1, g)} O_{X,x}[\varepsilon],$$

defines an element of $K_0((\mathcal{L}_{-2}(X[\varepsilon])/\mathcal{L}_{-3}(X[\varepsilon]))^\#)(\cong \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]))$,

with $\varepsilon^2 = 0$.

Theorem 3.2. $L' \bullet \in Z_2^M(D^{\text{Perf}}(X_1))$, i.e., $L' \bullet \in \text{Ker}(d_{1,X_1}^{2,-2})$.

Proof. Under the isomorphism in Theorem 1.4, for $j = 1$ and $i = -2$,

$$K_0((\mathcal{L}_{(-2)}(X[\varepsilon])/\mathcal{L}_{(-3)}(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]), \text{ with } \varepsilon^2 = 0,$$

$L' \bullet$ decomposes into the direct sum of

$$L'_1 \bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(\frac{g}{h}, -(f + \varepsilon \frac{w_1}{h}))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f + \varepsilon \frac{w_1}{h}, \frac{g}{h})} (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$L'_2 \bullet : 0 \rightarrow (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(\frac{g}{f}, -(h + \varepsilon \frac{w_1}{f}))^T} (O_{X,x})_{(h,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(h + \varepsilon \frac{w_1}{f}, \frac{g}{f})} (O_{X,x})_{(h,g)}[\varepsilon].$$

Since h^{-1} exists in $(O_{X,x})_{(f,g)}$ and f^{-1} exists in $(O_{X,x})_{(g,h)}$, the above two complexes are **quasi-isomorphic** to the following complexes respectively, still called $L'_1{}^\bullet$ and $L'_2{}^\bullet$ by abuse of notations,

$$L'_1{}^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g, -(f+\varepsilon \frac{w_1}{h}))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f+\varepsilon \frac{w_1}{h}, g)} (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$L'_2{}^\bullet : 0 \rightarrow (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(g, -(h+\varepsilon \frac{w_1}{f}))^T} (O_{X,x})_{(h,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(h+\varepsilon \frac{w_1}{f}, g)} (O_{X,x})_{(h,g)}[\varepsilon].$$

Noting $O_{X,y} = (O_{X,x})_{(f,g)}$, we have $L'_1{}^\bullet \in K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$. Similarly, $L'_2{}^\bullet \in K_0(O_{X,z}[\varepsilon] \text{ on } z[\varepsilon])$. Mimicking the algorithm by Green-Griffiths [3](page 131), the image of $L'_1{}^\bullet$ under the Chern map

$$\text{Chern} : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \rightarrow \bigoplus_{y \in X^{(2)}} H_y^2(\Omega_{X/\mathbb{Q}}^1),$$

may be described as follows. The following diagram, associated to $L'_1{}^\bullet$, (3.1)

$$\begin{cases} (O_{X,x})_{(f,g)} \xrightarrow{(g, -f)^T} (O_{X,x})_{(f,g)}^{\oplus 2} \xrightarrow{(f,g)} (O_{X,x})_{(f,g)} \longrightarrow (O_{X,x})_{(f,g)}/(f,g) \longrightarrow 0 \\ (O_{X,x})_{(f,g)} \xrightarrow{\frac{-w_1 dg}{h}} \Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1, \end{cases}$$

gives an element α in $\text{Ext}_{(O_{X,x})_{(f,g)}}^2((O_{X,x})_{(f,g)}/(f,g), \Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)$. Noting that

$$H_y^2(\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1) = \varinjlim_{n \rightarrow \infty} \text{Ext}_{(O_{X,x})_{(f,g)}}^2((O_{X,x})_{(f,g)}/(f,g)^n, \Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1),$$

the image $[\alpha]$ of α under the limit is in $H_y^2(\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)$ and it is the image of $L'_1{}^\bullet$ under the Chern map.

Similarly, the following diagram, associated to $L'_2{}^\bullet$, (3.2)

$$\begin{cases} (O_{X,x})_{(h,g)} \xrightarrow{(g, -h)^T} (O_{X,x})_{(h,g)}^{\oplus 2} \xrightarrow{(h,g)} (O_{X,x})_{(h,g)} \longrightarrow (O_{X,x})_{(h,g)}/(h,g) \longrightarrow 0 \\ (O_{X,x})_{(h,g)} \xrightarrow{\frac{-w_1 dg}{f}} \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1, \end{cases}$$

gives an element β in $\text{Ext}_{(O_{X,x})_{(h,g)}}^2((O_{X,x})_{(h,g)}/(h,g), \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1)$. Noting that

$$H_z^2(\Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1) = \varinjlim_{n \rightarrow \infty} \text{Ext}_{(O_{X,x})_{(h,g)}}^2((O_{X,x})_{(h,g)}/(h,g)^n, \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1),$$

the image $[\beta]$ of β under the limit is in $H_z^2(\Omega_{(O_{X,x})(h,g)/\mathbb{Q}}^1)$ and it is the image of L_2^\bullet under the Chern map.

Mimicking Green-Griffiths [3], page 103, $\partial_1^{2,-2}$ maps α in $H_x^3(\Omega_{X/\mathbb{Q}}^1)$ to :

$$(3.3) \quad \begin{cases} O_{X,x} \xrightarrow{M_1} O_{X,x}^{\oplus 3} \xrightarrow{M_2} O_{X,x}^{\oplus 3} \xrightarrow{M_3} O_{X,x} \longrightarrow O_{X,x}/(f,g,h) \longrightarrow 0 \\ O_{X,x} \xrightarrow{w_1 dg} \Omega_{O_{X,x}/\mathbb{Q}}^1, \end{cases}$$

where M_1, M_2 and M_3 are matrices associated to the Koszul resolution of $O_{X,x}/(f,g,h)$:

$$M_1 = \begin{pmatrix} f \\ -g \\ h \end{pmatrix}, M_2 = \begin{pmatrix} 0 & -h & -g \\ -h & 0 & f \\ g & f & 0 \end{pmatrix}, M_3 = (f, g, h).$$

Similarly, $\partial_1^{2,-2}$ maps β in $H_x^3(\Omega_{X/\mathbb{Q}}^1)$ to :

$$(3.4) \quad \begin{cases} O_{X,x} \xrightarrow{N_1} O_{X,x}^{\oplus 3} \xrightarrow{N_2} O_{X,x}^{\oplus 3} \xrightarrow{N_3} O_{X,x} \longrightarrow O_{X,x}/(h,g,f) \longrightarrow 0 \\ O_{X,x} \xrightarrow{w_1 dg} \Omega_{O_{X,x}/\mathbb{Q}}^1, \end{cases}$$

where N_1, N_2 and N_3 are matrices associated to the Koszul resolution of $O_{X,x}/(h,g,f)$:

$$N_1 = \begin{pmatrix} h \\ -g \\ f \end{pmatrix}, N_2 = \begin{pmatrix} 0 & -f & -g \\ -f & 0 & h \\ g & h & 0 \end{pmatrix}, N_3 = (h, g, f).$$

Mimicking the argument by Green-Griffiths [3], page 103, and noting the commutative diagram below

$$\begin{array}{ccccccccc} O_{X,x} & \xrightarrow{M_1} & O_{X,x}^{\oplus 3} & \xrightarrow{M_2} & O_{X,x}^{\oplus 3} & \xrightarrow{M_3} & O_{X,x} & \longrightarrow & O_{X,x}/(f,g,h) & \longrightarrow & 0 \\ -1 \downarrow & & W_1 \downarrow & & W_2 \downarrow & & 1 \downarrow & & \cong \downarrow & & \\ O_{X,x} & \xrightarrow{N_1} & O_{X,x}^{\oplus 3} & \xrightarrow{N_2} & O_{X,x}^{\oplus 3} & \xrightarrow{N_3} & O_{X,x} & \longrightarrow & O_{X,x}/(h,g,f) & \longrightarrow & 0, \end{array}$$

where W_1 and W_2 stand for the following matrices:

$$W_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, W_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

one can see that $\partial_1^{2,-2}(\alpha)$ and $\partial_1^{2,-2}(\beta)$ are negative of each other in $Ext_{O_{X,x}}^3(O_{X,x}/(f,g,h), \Omega_{O_{X,x}/\mathbb{Q}}^1)$. Hence, $\partial_1^{2,-2}(\alpha + \beta)$ is 0 in $H_x^3(\Omega_{X/\mathbb{Q}}^1)$.

Therefore, $d_{1,X_1}^{2,-2}(L^\bullet) = 0$ because of the commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{y \in X^{(2)}} H_y^2(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow{\text{Chern}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\
\partial_1^{2,-2} \downarrow & & d_{1,X_1}^{2,-2} \downarrow \\
\bigoplus_{x \in X^{(3)}} H_x^3(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(3)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]).
\end{array}$$

□

Theorem 3.3. [7] *For X a nonsingular projective 3-fold over a field k of characteristic 0 and for $j = 2$ and $q = 2$ in Theorem 3.14 in [7], we have the following commutative diagram (For $w_i = x_i, y_i, z_i$ below with $i = 1$ and 2, $K_*^M(O_{X_i, w_i} \text{ on } w_i) \cong K_*(O_{X_i, w_i} \text{ on } w_i)_{\mathbb{Q}}$. This may be explained (in a similar way) by Lemma 2.1. We omit the subscript \mathbb{Q} in the diagram below), the left arrows are induced by Chern character from K -theory to negative cyclic homology:*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(\Omega_{k(X)/\mathbb{Q}}^1)^{\oplus 2} & \xleftarrow{\text{Chern}} & K_2^M(k(X_2)) & \xrightarrow{\varepsilon=0} & K_2^M(k(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{z \in X^{(1)}} H_z^1((\Omega_{X/\mathbb{Q}}^1)^{\oplus 2}) & \xleftarrow{\text{Chern}} & \bigoplus_{z_2 \in X_2^{(1)}} K_1(O_{X_2, z_2} \text{ on } z_2) & \xrightarrow{\vdots} & \bigoplus_{z \in X^{(1)}} K_1(O_{X, z} \text{ on } z) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{y \in X^{(2)}} H_y^2((\Omega_{X/\mathbb{Q}}^1)^{\oplus 2}) & \xleftarrow{\text{Chern}} & \bigoplus_{y_2 \in X_2^{(1)}} K_0(O_{X_2, y_2} \text{ on } y_2) & \longrightarrow & \bigoplus_{y \in X^{(1)}} K_0(O_{X, y} \text{ on } y) \\
(\partial_1^{2,-2})^{\oplus 2} \downarrow & & d_{1, X_2}^{2,-2} \downarrow & & d_{1, X}^{2,-2} \downarrow \\
\bigoplus_{x \in X^{(3)}} H_x^3((\Omega_{X/\mathbb{Q}}^1)^{\oplus 2}) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x_2 \in X_2^{(3)}} K_{-1}(O_{X_2, x_2} \text{ on } x_2) & \longrightarrow & \bigoplus_{x \in X^{(3)}} K_{-1}(O_{X, x} \text{ on } x) = 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

Now, we consider $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)$, where w_1, w_2 are arbitrary elements of $O_{X,x}$ and $\varepsilon^3 = 0$. The Koszul resolution of $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)$,

$$L''^\bullet : 0 \rightarrow O_{X,x}[\varepsilon] \xrightarrow{(g, -(fh + \varepsilon w_1 + \varepsilon^2 w_2))^T} O_{X,x}[\varepsilon]^{\oplus 2} \xrightarrow{(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)} O_{X,x}[\varepsilon],$$

defines an element of $K_0(\mathcal{L}_{-2}(X[\varepsilon])/\mathcal{L}_{-3}(X[\varepsilon]))^\# (\cong \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]))$,

with $\varepsilon^3 = 0$.

Under the isomorphism in Theorem 1.4, for $j = 2$ and $i = -2$,

$$K_0((\mathcal{L}_{(-2)}(X[\varepsilon])/\mathcal{L}_{(-3)}(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]), \text{ with } \varepsilon^3 = 0,$$

L''^\bullet decomposes into the direct sum of

$$L_1''^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(\frac{g}{h}, -(f+\varepsilon\frac{w_1}{h} + \varepsilon^2\frac{w_2}{h}))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f+\varepsilon\frac{w_1}{h} + \varepsilon^2\frac{w_2}{h}, \frac{g}{h})} (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$L_2''^\bullet : 0 \rightarrow (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(\frac{g}{f}, -(h+\varepsilon\frac{w_1}{f} + \varepsilon^2\frac{w_2}{f}))^T} (O_{X,x})_{(h,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(h+\varepsilon\frac{w_1}{f} + \varepsilon^2\frac{w_2}{f}, \frac{g}{f})} (O_{X,x})_{(h,g)}[\varepsilon].$$

Since h^{-1} exists in $(O_{X,x})_{(f,g)}$, $L_1''^\bullet$ is quasi-isomorphic to the following complex, still called $L_1''^\bullet$ by abuse of notations,

$$L_1''^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g, -(f+\varepsilon\frac{w_1}{h} + \varepsilon^2\frac{w_2}{h}))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f+\varepsilon\frac{w_1}{h} + \varepsilon^2\frac{w_2}{h}, g)} (O_{X,x})_{(f,g)}[\varepsilon].$$

Noting $O_{X,y} = (O_{X,x})_{(f,g)}$, we have $L_1''^\bullet \in K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$. The image of $L_1''^\bullet$ under the Chern map

$$\text{Chern} : \bigoplus_{y_2 \in X_2^{(2)}} K_0(O_{X_2,y_2} \text{ on } y_2) \rightarrow \bigoplus_{y \in X^{(2)}} H_y^2((\Omega_{X/\mathbb{Q}}^1)^{\oplus 2})$$

may be described similarly as the Chern map in Theorem 3.2.

The following diagram, associated to $L_1''^\bullet$,

$$(3.5) \quad \begin{array}{ccccccc} \begin{cases} (O_{X,x})_{(f,g)} \\ (O_{X,x})_{(f,g)} \end{cases} & \xrightarrow{\begin{smallmatrix} (g, -f)^T \\ -\frac{w_1 dg}{h} - \frac{w_2 dg}{h} \end{smallmatrix}} & (O_{X,x})_{(f,g)}^{\oplus 2} & \xrightarrow{(f,g)} & (O_{X,x})_{(f,g)} & \longrightarrow & (O_{X,x})_{(f,g)}/(f,g) \longrightarrow 0 \\ & & & & & & \end{array}$$

gives an element α in $\text{Ext}_{(O_{X,x})_{(f,g)}}^2((O_{X,x})_{(f,g)}/(f,g), (\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)^{\oplus 2})$.

Noting that

$$H_y^2((\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)^{\oplus 2}) = \varinjlim_{n \rightarrow \infty} \text{Ext}_{(O_{X,x})_{(f,g)}}^2((O_{X,x})_{(f,g)}/(f,g)^n, (\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)^{\oplus 2}),$$

the image $[\alpha]$ of α under the limit is in $H_y^2((\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)^{\oplus 2})$ and it is the image of $L_1''^\bullet$ in $H_y^2((\Omega_{(O_{X,x})_{(f,g)}/\mathbb{Q}}^1)^{\oplus 2})$. We have the similar description of $L_2''^\bullet$. By repeating the argument in Theorem 3.2, we can show

Theorem 3.4. $L''^\bullet \in Z_2^M(D^{\text{Perf}}(X_2))$, i.e., $L''^\bullet \in \text{Ker}(d_{1,X_2}^{2,-2})$.

Now, by taking $w_1 = f_1 h$, the Koszul complex L'^\bullet in Theorem 3.2 is of the form

$$(3.6) \quad L'^\bullet : 0 \rightarrow O_{X,x}[\varepsilon] \xrightarrow{(g, -(fh + \varepsilon f_1 h))^T} O_{X,x}[\varepsilon]^{\oplus 2} \xrightarrow{(fh + \varepsilon f_1 h, g)} O_{X,x}[\varepsilon].$$

Under the isomorphism in Theorem 1.4, the complex L'^\bullet decomposes into the direct sum of $L'_1{}^\bullet$ and $L'_2{}^\bullet$,

$$(3.7) \quad L'^\bullet = L'_1{}^\bullet + L'_2{}^\bullet,$$

where $L'_1{}^\bullet$ and $L'_2{}^\bullet$ are of the forms

$$(3.8) \quad L'_1{}^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g, -(f + \varepsilon f_1))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f + \varepsilon f_1, g)} (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$(3.9) \quad L'_2{}^\bullet : 0 \rightarrow (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(g, -(h + \varepsilon \frac{f_1 h}{f}))^T} (O_{X,x})_{(h,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(h + \varepsilon \frac{f_1 h}{f}, g)} (O_{X,x})_{(h,g)}[\varepsilon].$$

Now, by taking $w_1 = f_1 h$ and $w_2 = f_2 h$, where f_2 is an arbitrary element of $O_{X,x}$, the Koszul complex L''^\bullet in Theorem 3.4 is of the form

$$(3.10) \quad L''^\bullet : 0 \rightarrow O_{X,x}[\varepsilon] \xrightarrow{(g, -(fh + \varepsilon f_1 h + \varepsilon^2 f_2 h))^T} O_{X,x}[\varepsilon]^{\oplus 2} \xrightarrow{(fh + \varepsilon f_1 h + \varepsilon^2 f_2 h, g)} O_{X,x}[\varepsilon].$$

Under the isomorphism in Theorem 1.4, the complex L''^\bullet decomposes into the direct sum of $L''_1{}^\bullet$ and $L''_2{}^\bullet$,

$$(3.11) \quad L''^\bullet = L''_1{}^\bullet + L''_2{}^\bullet,$$

where $L''_1{}^\bullet$ and $L''_2{}^\bullet$ are of the forms

$$(3.12) \quad L''_1{}^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g, -(f + \varepsilon f_1 + \varepsilon^2 f_2))^T} (O_{X,x})_{(f,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(f + \varepsilon f_1 + \varepsilon^2 f_2, g)} (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$(3.13) \quad L''_2{}^\bullet : 0 \rightarrow (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(g, -(h + \varepsilon \frac{f_1 h}{f} + \varepsilon^2 \frac{f_2 h}{f}))^T} (O_{X,x})_{(h,g)}[\varepsilon]^{\oplus 2} \xrightarrow{(h + \varepsilon \frac{f_1 h}{f} + \varepsilon^2 \frac{f_2 h}{f}, g)} (O_{X,x})_{(h,g)}[\varepsilon].$$

Lemma 3.5. $L'_2{}^\bullet \in Z_2^M(D^{\text{Perf}}(X_1))$ and $L''_2{}^\bullet \in Z_2^M(D^{\text{Perf}}(X_2))$

Proof. The proof can be done by mimicking the argument in Theorem 3.2. We sketch it for readers' convenience.

The following diagram, associated to L_2^\bullet ,
(3.14)

$$\begin{cases} (O_{X,x})_{(h,g)} \xrightarrow{(g,-h)^T} (O_{X,x})_{(h,g)}^{\oplus 2} \xrightarrow{(h,g)} (O_{X,x})_{(h,g)} \longrightarrow (O_{X,x})_{(h,g)}/(h,g) \longrightarrow 0 \\ (O_{X,x})_{(h,g)} \xrightarrow{\frac{-f_1 h d g}{f}} \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1 \end{cases}$$

gives an element β in $Ext_{(O_{X,x})_{(h,g)}}^2((O_{X,x})_{(h,g)}/(h,g), \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1)$. Noting $\frac{f_1 h d g}{f} = h \frac{f_1 d g}{f}$, $\beta \equiv 0 \in Ext_{(O_{X,x})_{(h,g)}}^2((O_{X,x})_{(h,g)}/(h,g), \Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1)$. Hence the image $[\beta]$ of β in $H_z^2(\Omega_{(O_{X,x})_{(h,g)}/\mathbb{Q}}^1)$ is 0. In other words, the image of L_1^\bullet is 0 under the Chern map. It is trivial that $\partial_1^{2,-2}([\beta]) = 0$ in $H_x^3(\Omega_{X/\mathbb{Q}}^1)$.

Therefore, $d_{1,X_1}^{2,-2}(L_1^\bullet) = 0$ because of the commutative diagram:

$$\begin{array}{ccc} \bigoplus_{y \in X^{(2)}} H_y^2(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow{\text{Chern}} & \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\ \partial_1^{2,-2} \downarrow & & d_{1,X_1}^{2,-2} \downarrow \\ \bigoplus_{x \in X^{(3)}} H_x^3(\Omega_{X/\mathbb{Q}}^1) & \xleftarrow[\cong]{\text{Chern}} & \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(3)}} K_{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]). \end{array}$$

Similar argument works for $L_2^{\bullet\bullet}$. □

Let L^\bullet denote the following complex, which is the Koszul resolution of $O_{X,x}/(fh, g)$

$$(3.15) \quad L^\bullet : 0 \rightarrow O_{X,x} \xrightarrow{(g,-fh)^T} O_{X,x}^{\oplus 2} \xrightarrow{(fh,g)} O_{X,x}.$$

Under the isomorphism in Theorem 1.4, L^\bullet decomposes into the direct sum of L_1^\bullet and L_2^\bullet , which are (quasi-isomorphic to) of the forms

$$L_1^\bullet : 0 \rightarrow (O_{X,x})_{(f,g)} \xrightarrow{(g,-f)^T} (O_{X,x})_{(f,g)}^{\oplus 2} \xrightarrow{(f,g)} (O_{X,x})_{(f,g)},$$

and

$$L_2^\bullet : 0 \rightarrow (O_{X,x})_{(g,h)} \xrightarrow{(g,-h)^T} (O_{X,x})_{(g,h)}^{\oplus 2} \xrightarrow{(h,g)} (O_{X,x})_{(g,h)}.$$

Theorem 3.6. *The answer to TingFai Ng's Question 2.4 is positive:*

We can take $\gamma = L^\bullet(3.15) \in Z_2^M(D^{\text{Perf}}(X))$, it satisfies that $L^\bullet = L_1^\bullet + L_2^\bullet$, where $L_1^\bullet = \mu(Y)$ and $L_2^\bullet = \mu(Z)$. The complex $L^\bullet(3.6)$ is a first order deformation of γ , satisfying:

1. $L^\bullet = L_1^\bullet + L_2^\bullet(3.7)$, where in fact $L_1^\bullet = \mu(Y')$ and $L_2^\bullet = \mu(Z')$, with Z' a first order infinitesimal deformation of Z , which is generically

given by $(h + \varepsilon \frac{f_1 h}{f}, g) \subset O_{X,z}[\varepsilon] (= (O_{X,x})_{(h,g)}[\varepsilon])$, with $\varepsilon^2 = 0$ (please note $(h + \varepsilon \frac{f_1 h}{f}, g) \not\subset O_{X,x}[\varepsilon]$, since $\frac{f_1 h}{f} \notin O_{X,x}$).

2. the first order deformation given by $L'^\bullet(3.6)$ extends to second order $L''^\bullet(3.10) \in Z_2^M(D^{\text{Perf}}(X_2))$, and

3. the first order deformation given by $\mu(Z')$ extends to second order $L_2''^\bullet(3.13) \in Z_2^M(D^{\text{Perf}}(X_2))$.

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